The Power of Derivatives and Related Rates

Consider a circle of radius r increasing at a certain rate with respect to time t. This implies that the radius of the circle is also increasing with respect to time. The following model will reveal a relationship between quantities.

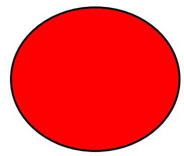
Circle

Area- The area of the circle can be represented by the equation $A = \pi r^2$. Differentiating both sides of the equation with respect to t gives us the following **Related Rates Equation.**

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) = \pi \frac{d}{dt}(r^2)$$ Differentiating both sides with respect to t.

$$\frac{dA}{dt} = \pi 2r \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt},$$

(1) $$\boxed{\frac{dA}{dt} = 2\pi r \frac{dr}{dt}}$$ **Related Rates Equation**



The change in area with respect to time is related to the change in radius with respect to time.

Circumference- The circumference of a circle can be represented by the equation $C = 2\pi r$. Differentiating both sides with respect to t gives us the following **Related Rates Equation.**

$$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r) = 2\pi \frac{d}{dt}(r)$$ Differentiating both sides with respect to t.

(2) $$\boxed{\frac{dC}{dt} = 2\pi \frac{dr}{dt}}$$ **Related Rates Equation**

The change in circumference with respect to time is related to the change in radius with respect to time.

Typical Problems

Q ***The area of a circle is increasing at the rate of 12 inches per minute, when the radius of the circle measures 4 inches, what is the change in radius with respect to time?***

Answer-

$$12 = 2\pi 4 \frac{dr}{dt}$$ Using formula (1)

$$12 = 8\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{12}{8\pi} \Rightarrow \frac{dr}{dt} = \frac{3}{2\pi}$$

Q ***The radius of a circle is decreasing at the rate of 2 inches per second, when the circle has a radius of 6 inches, what is the rate of change in the area with respect to time?***

Answer-

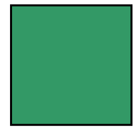
$$-2 = 2\pi 6 \frac{dr}{dt}$$ Using formula (2)

$$-2 = 12\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-1}{6\pi}$$ inches per second

Square

Area- The area of a square can be represented by the equation $A = lw$. Differentiating both sides of the equation with respect to t gives us the following **Related Rates Equation.**

$$\frac{d}{dt}(A) = \frac{d}{dt}(lw) = w\frac{dl}{dt} + l\frac{dw}{dt} \Rightarrow$$



(3) $\boxed{\frac{dA}{dt} = w\frac{dl}{dt} + l\frac{dw}{dt}}$ **Related Rates Equation**

The change in area with respect to time is related to the change in length, and the change in height, with respect to time.

Perimeter- The perimeter of a square can be represented by the equation $P = 2l + 2w$. Differentiating both sides with respect to t gives us the following **Related Rates Equation.**

$$\frac{d}{dt}(P) = \frac{d}{dt}(2l + 2w) = \frac{d}{dt}(2l) + \frac{d}{dt}(2w) = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

(4) $\boxed{\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}}$ **Related Rates Equation**

The change in Perimeter with respect to time is related to the change in length, and the change in width, with respect to time.

Typical Problems

Q ***The area of a rectangle is increasing at the rate of 55 inches per minute, when the length of the rectangle measures 5 inches and is increasing at a rate of 2 inches per second, while the width of the rectangle measures 11 inches, what is the change in width with respect to time?***

Answer-

$$55 = 5 \cdot 2 + 11\frac{dw}{dt} \text{ Using formula (3)}$$

$$55 = 10 + 11\frac{dw}{dt} \Rightarrow 11\frac{dw}{dt} = 45 \Rightarrow \frac{dw}{dt} = \frac{45}{11} \text{ inches per minute}$$

Q ***What is the rate of change for the perimeter of a square, if the length is 12 cm and is decreasing at the rate of 4 cm per minute, and the width is 6 cm and is increasing at the rate of 3 cm per second?***

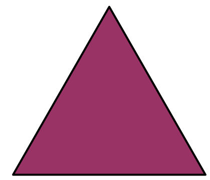
Answer-

$$\frac{dP}{dt} = 2(-4) + 2 \cdot 3 \text{ Using Formula (4)}$$

$$\frac{dP}{dt} = -2 \text{ cm per minute}$$

Triangle

Area- The area of the triangle is represented by the equation $A = \frac{1}{2}bh$. Differentiating both sides of the equation with respect to t gives us the following **Related Rates Equation.**



$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2}bh\right) = \frac{1}{2}\frac{d}{dt}(bh)$$

$$\frac{dA}{dt} = \frac{1}{2}\left[h\frac{db}{dt} + b\frac{dh}{dt}\right] \Rightarrow$$

(5) $\boxed{\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt} + \frac{1}{2}\frac{dh}{dt}}$ **Related Rates Equation**

The change in area with respect to time is related to the change in base, and change in height, with respect to time.

Typical Problems

Q ***The base of a triangle is decreasing at the rate of –3 inches per second, while the height of the triangle is increasing at the rate of 6 inches per second, what is the rate of increase, or decrease, in the area of the triangle?***

Answer-

$$\frac{dA}{dt} = \frac{1}{2}(-3) + \frac{1}{2}(9) \text{ Using formula (5)}$$

$$\frac{dA}{dt} = \frac{1}{2}(-3+9) = \frac{1}{2}6 = 3 \text{ inches per second}$$

Q ***The area of a triangle is increasing at the rate of 12 meters per second, while the height of the triangle remains constant, what is the rate of increase, or decrease, of the base?***

Answer-

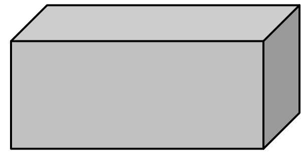
$$12 = \frac{1}{2}(0) + \frac{1}{2}\frac{dh}{dt} \text{ Using Formula (5)}$$

$$12 = \frac{1}{2}\frac{dh}{dt} \Rightarrow$$

$$\frac{dh}{dt} = 24 \text{ meters per second}$$

Cube

Volume- The volume of a cube is represented by $V = lwh$. Differentiating both sides with respect to t gives the following **Related Rates Equation**.



$$\frac{d}{dt}(V) = \frac{d}{dt}(lwh)$$

$$\frac{dV}{dt} = \frac{d}{dt}(lwh) = wh\frac{d}{dt}(l) + l\frac{d}{dt}(wh)$$ by the product rule

$$\frac{dV}{dt} = wh\frac{d}{dt}(l) + l\left[w\frac{d}{dt}(h) + h\frac{d}{dt}(w)\right]$$

(6) $$\boxed{\frac{dV}{dt} = wh\frac{dl}{dt} + lw\frac{dh}{dt} + lh\frac{dw}{dt}}$$ **Related Rates Equation**

The change in volume with respect to time is related to the change in length, the change in height, and the change in width with respect to time.

Surface Area- The surface area of a cube is represented by $S = 2lw + 2lh + 2hw$. Differentiating both sides with respect to t gives the following **Related Rates Equation**.

$$\frac{d}{dt}(S) = \frac{d}{dt}(2lw + 2lh + 2hw)$$

$$\frac{dS}{dt} = \frac{d}{dt}(2lw) + \frac{d}{dt}(2lh) + \frac{d}{dt}(2hw) = 2\frac{d}{dt}(lw) + 2\frac{d}{dt}(lh) + 2\frac{d}{dt}(hw)$$

$$\frac{dS}{dt} = 2\left[\frac{d}{dt}(lw) + \frac{d}{dt}(lh) + \frac{d}{dt}(hw)\right]$$

$$\frac{dS}{dt} = 2\left[l\frac{d}{dt}(w) + w\frac{d}{dt}(l) + l\frac{d}{dt}(h) + h\frac{d}{dt}(l) + w\frac{d}{dt}(h) + h\frac{d}{dt}(w)\right]$$

$$\frac{dS}{dt} = 2\left[(l+h)\frac{dw}{dt} + (w+h)\frac{dl}{dt} + (w+l)\frac{dh}{dt}\right]$$

(7) $$\boxed{\frac{dS}{dt} = 2(l+h)\frac{dw}{dt} + 2(w+h)\frac{dl}{dt} + 2(w+l)\frac{dh}{dt}}$$ **Related Rates Equation**

The change in surface area with respect to time is related to the change in length, the change in height, and the change in width with respect to time.

Typical Problems

Q ***The volume of a cube is increasing at the rate of 49 feet per minute, while the length is decreasing at the rate of 12 feet per minute, the width is increasing at the rate of 8 feet per minute, and the height is decreasing at the rate of 2 feet per minute. When the triangle measures 45 feet (length) by 30 feet (width) by 20 feet (height), determine the rate of increase, or decrease, of the volume with respect to time?***

Answer-

$$\frac{dV}{dt} = 30 \cdot 20 \cdot (-12) + 45 \cdot 30 \cdot (-2) + 45 \cdot 20 \cdot 8 \text{ Using formula (6)}$$

$$\frac{dV}{dt} = -7200 - 2700 + 7200$$

$$\frac{dV}{dt} = -2700 \text{ feet per minute}$$

Q ***Using the same conditions above, determine the change in surface area with respect to time.***

Answer-

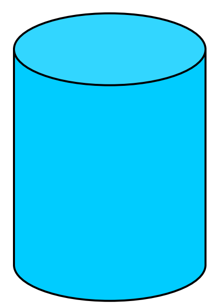
$$\frac{dS}{dt} = 2(45+20)8 + 2(30+20)(-12) + 2(30+45)(-2) \text{ Using formula (7)}$$

$$\frac{dS}{dt} = 1040 - 1200 + -300$$

$$\frac{dS}{dt} = -460 \text{ feet per minute}$$

Cylinder

Volume- The volume of a cylinder is represented by $V = \pi r^2 h$. Differentiating both sides with respect to t gives the following **Related Rates Equation**.



$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h)$$

$$\frac{dV}{dt} = \pi \frac{d}{dt}(r^2 h) = \pi \left[h \frac{d}{dt}(r^2) + r^2 \frac{d}{dt}(h) \right]$$

$$\frac{dV}{dt} = \pi \left[hr \frac{dr}{dt} + r^2 h \right]$$

$$(8) \quad \boxed{\frac{dV}{dt} = \pi hr \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}} \text{ Related Rates Equation}$$

The change in volume with respect to time is related to the change in radius, and the change in height, with respect to time.

Surface Area- The surface area of a cylinder is represented by $S = 2\pi r^2 + 2\pi rh$. Differentiating both sides with respect to t gives the following **Related Rates Equation**.

$$\frac{d}{dt}(S) = \frac{d}{dt}(2\pi r^2 + 2\pi rh)$$

$$\frac{dS}{dt} = \frac{d}{dt}(2\pi r^2) + \frac{d}{dt}(2\pi rh) = 2\pi \frac{d}{dt}(r^2) + 2\pi \frac{d}{dt}(rh)$$

$$\frac{dS}{dt} = 2\pi 2r\frac{d}{dt}(r) + 2\pi\left[r\frac{d}{dt}(h) + h\frac{d}{dt}(r)\right]$$

$$\frac{dS}{dt} = 4\pi r\frac{dr}{dt} + 2\pi\left[r\frac{dh}{dt} + h\frac{dr}{dt}\right] = 4\pi r\frac{dr}{dt} + 2\pi r\frac{dh}{dt} + 2\pi h\frac{dr}{dt}$$

$$\frac{dS}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$$

$$\frac{dS}{dt} = 2\pi(2r + h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$$

(9) $\boxed{\frac{dS}{dt} = 2\pi\left[(2r + h)\frac{dr}{dt} + r\frac{dh}{dt}\right]}$ **Related Rates Equation**

The change in surface area with respect to time is related to the change in radius, and the change in height, with respect to time.

Typical Problems

Q ***The volume of a cylinder is decreasing at the rate of 3 feet per hour, while the height is increasing at the rate of 3/pi feet per hour, and the radius is decreasing at the rate of –2/pi feet per hour. What is the radius of the cylinder, when the height is 4 feet?***

Answer-

$$3 = \pi 4r\frac{-2}{\pi} + \pi r^2\frac{3}{\pi}$$ Using equation (8)

$$3 = -8r + 3r^2 \Rightarrow 3r^2 - 8r - 3 = 0$$

$$\Rightarrow (3r + 1)(r - 3) = 0$$

$$\Rightarrow r = \frac{-1}{3}, r = 3$$

Thus the radius is 3 feet.

Q ***The radius of a cylinder is 12 cm, while the height of the cylinder is 45 cm. If the radius of the cylinder is increasing at the rate of 4 cm per day, and the height remains constant (no change), then what is the rate of increase, or decrease, in the surface area of the cylinder?***

Answer-

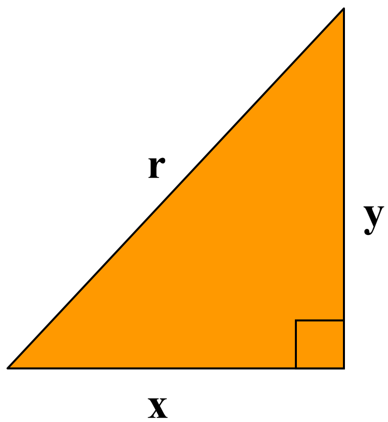
$$\frac{dS}{dt} = 2\pi\left[(2\cdot 12 + 45)4 + 12\cdot 0\right]$$ Using equation (9)

$$\frac{dS}{dt} = 2\pi[24 + 45] = 2\pi 69 = 138\pi$$

$$\frac{dS}{dt} = 138\pi \text{ cm per day}$$

Right Triangle

This shape is of particular interest to calculus students because many natural phenomena are modeled with a **Right Triangle**.



Right Triangle Facts

- $x^2 + y^2 = r^2$
- $\cos\ \theta\ = \frac{x}{r}$
- $\sin\ \theta\ = \frac{y}{\ }$
- n $\theta\ = \frac{y}{x}$

We can derive various related rate equations by differentiating with respect to time t. This will provide us with our **Related Rates Equations**.

Pythagorean Theorem: $x^2 + y^2 = r^2$

$$\frac{d}{dt}\left(x^2+y^2\right)=\frac{d}{dt}\left(r^2\right)\Rightarrow\frac{d}{dt}\left(x^2\right)+\frac{d}{dt}\left(y^2\right)=\frac{d}{dt}\left(r^2\right)$$

$$\Rightarrow 2x\frac{d}{dt}(x)+2y\frac{d}{dt}(y)=2r\frac{d}{dt}(r)$$

$$\Rightarrow \ (10)\ \boxed{x\frac{dx}{dt}+y\frac{dy}{dt}=r\frac{dr}{dt}}$$

The change in r with respect to time is related to the change in x, and the change in y, with respect to time.

Definition of Cosine: $\cos(\theta)=\frac{x}{r}$

$$x=r\cos(\theta)\Rightarrow\frac{d}{dt}(x)=\frac{d}{dt}\left[r\cos(\theta)\right]$$

$$\Rightarrow\frac{dx}{dt}=r\frac{d}{dt}\left[\cos(\theta)\right]+\cos(\theta)\frac{d}{dt}(r)$$

$$\Rightarrow (11)\ \boxed{\frac{dx}{dt}=-r\sin(\theta)\frac{d\theta}{dt}+\cos(\theta)\frac{dr}{dt}}$$

The change in x with respect to time is related to the change in theta, and the change in r, with respect to time.

Definition of Sine: $\sin(\theta) = \frac{y}{r}$

$$y = r\sin(\theta) \Rightarrow \frac{d}{dt}(y) = \frac{d}{dt}\left[r\sin(\theta)\right]$$

$$\Rightarrow \frac{dy}{dt} = r\frac{d}{dt}\left[\sin(\theta)\right] + \sin(\theta)\frac{d}{dt}(r)$$

$$\Rightarrow (12)\ \boxed{\frac{dy}{dt} = r\cos(\theta)\frac{d\theta}{dt} + \sin(\theta)\frac{dr}{dt}}$$

The change in y with respect to time is related to the change in theta, and the change in r, with respect to time.

Definition of Tangent: $\tan(\theta) = \frac{y}{x}$

$$y = x\tan(\theta) \Rightarrow \frac{d}{dt}(y) = \frac{d}{dt}\left[x\tan(\theta)\right]$$

$$\Rightarrow \frac{dy}{dt} = x\frac{d}{dt}\left[\tan(\theta)\right] + \tan(\theta)\frac{d}{dt}(x)$$

$$\Rightarrow (13)\ \boxed{\frac{dy}{dt} = x\sec^2(\theta)\frac{d\theta}{dt} + \tan(\theta)\frac{dx}{dt}}$$

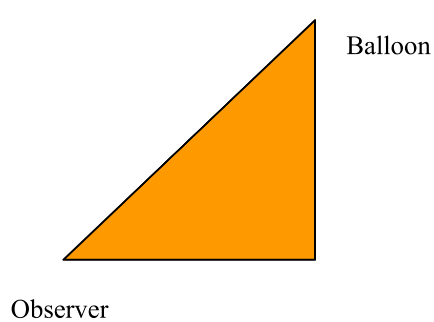
The change in y with respect to time is related to the change in theta, and the change in x, with respect to time.

Typical Related Rate Problems

Q ***A hot air balloon is rising straight up from the ground with an observer standing 1200 feet away from the point of lift off. When the observer measures an angle of pi/4 radians, he also measures the angle increasing at the rate of 0.23 radians per second. How fast is the balloon rising at the moment he makes his measurements?***

Answer- Using a right triangle, we obtain the following model.

$$x = 1200$$
$$\theta = \frac{\pi}{4}$$
$$\frac{d\theta}{dt} = 0.23$$
$$\frac{dx}{dt} = 0$$



To determine $\frac{dy}{dt}$ using (13) $\frac{dy}{dt} = x\sec^2(\theta)\frac{d\theta}{dt} + \tan(\theta)\frac{dx}{dt}$.

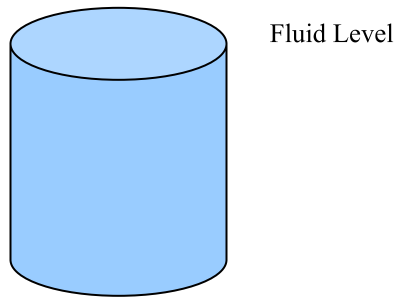
$$\frac{dy}{dt} = 1200\left[\sec^2\left(\frac{\pi}{4}\right)\right]0.23 + \left[\tan\left(\frac{\pi}{4}\right)\right]0$$

$$\frac{dy}{dt} = 1200 \cdot\cdot 0.23$$

Q ***How rapidly will the fluid level inside a vertical cylindrical tank with a radius of 10 meters drop, if we pump the fluid out at the rate of 2500 L/Hour?***
Answer- Using a cylinder, we obtain the following model.

To determine $\frac{dh}{dt}$ by using (8) $\frac{dV}{dt} = \pi hr\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

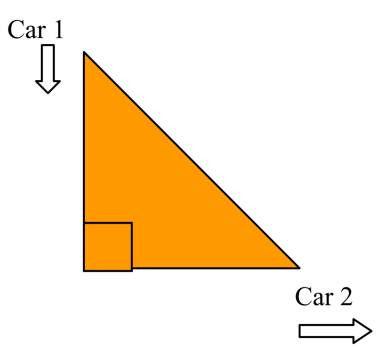
$$r = 10$$
$$\frac{dr}{dt} = 0$$
$$\frac{dV}{dt} = -2500$$



$$-2500 = \pi h 10 \cdot 0 + \pi \cdot 10^2\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{-2500}{100\pi}$$
$$\frac{dh}{dt} = -\frac{25}{\pi}$$

Q ***A car is approaching a right-angled intersection from the north, while a second car has made a right turn on the intersession and is now heading due east. If the first car that's heading south is 0.8 miles from the intersection, traveling at the rate of 45 miles per hour, while the distance between the two cars is decreasing at the rate of 10 miles per hour, what is the speed of the second car, as it's 0.6 miles from the intersection?***
Answer- Using a right triangle, we obtain the following model.

To determine $\frac{dx}{dt}$ by using (10) $\boxed{x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}}$



$$x = 0.6$$
$$= 0.8$$
$$\frac{dy}{dt} \quad 45$$
$$\frac{dr}{dt} =$$

$$0.6\frac{dx}{dt}+0.8(-45)=1\cdot(10)$$

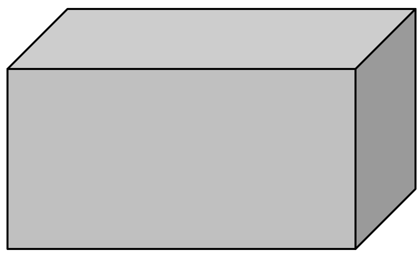
$$0.6\frac{dx}{dt}-3.6=10 \Rightarrow 0.6\frac{dx}{dt}=14.6 \Rightarrow \frac{dx}{dt}=\frac{14.6}{0.6}$$

$$\frac{dx}{dt}=24.33$$

Q ***Fluid in a rectangular tank is decreasing at the rate of 250 liters per hour. If the fluid in the tank measures 30 by 15 by 10 (LxWxH) meters, determine the rate of decrease in terms of height.***
Answer- Using a rectangular box, we obtain the following model.

To determine $\frac{dh}{dt}$ by using (6) $\boxed{\frac{dV}{dt}=wh\frac{dl}{dt}+lw\frac{dh}{dt}+lh\frac{dw}{dt}}$

$$l=30$$
$$w=15$$
$$h=10$$
$$\frac{dV}{dt}=-250$$
$$\frac{dl}{dt}=\frac{dw}{dt}=0$$



$$-250=15\cdot 10\cdot 0+30\cdot 15\cdot \frac{dh}{dt}+30\cdot 10\cdot 0$$

$$-250=450\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=-\frac{250}{450}$$

$$\frac{dh}{dt}=-0.56$$