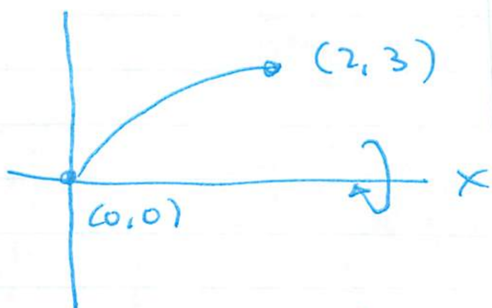
Surface Area

The Surface area S of a Solid generated by revolving a smooth curve C about the x-axis represented by $x = f(t)$ $y = g(t)$ $a \le t \le b$ where $y(t) \ge 0$ (above x axis)

$$S = \int_a^b 2\pi \sqrt{(x')^2 + (y')^2}\; dt$$

ex $x = 2t^3$; $y = 3t^2$; $0 \le t \le 1$ about x-axis



$$SA = 2\pi \int_0^1 3t^2 \sqrt{(6t^2)^2 + (6t)^2}\; dt$$

$$SA = 2\pi \int_0^1 3t^2 \sqrt{36t^4 + 36t^2}\; dt$$

$$= 2\pi \int_0^1 3t^2 \sqrt{36t^2(t^2+1)}\; dt$$

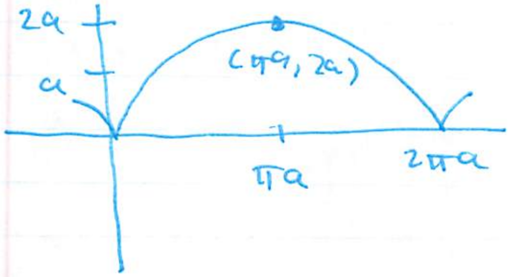
$$= 36\pi \int_0^1 t^3 \sqrt{t^2+1}\; dt$$; $u = t^2 + 1$, $du = 2t\,dt$

$$= \frac{36\pi}{2} \int_1^2 (u-1)\sqrt{u}\; du = 18\pi \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \Big|_1^2 \right]$$

$$= \frac{24\pi}{5} (\sqrt{2} + 1)$$

* Find the arc length of a cycloid

$x = a(t - \sin t)$; $y = a(1 - \cos t)$ $\quad a > 0$



one arch of cycloid
$0 < t < 2\pi$

note the curve is smooth $0 < t < 2\pi$

$$s = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\ dt$$

$$\sqrt{a^2\left[(1-\cos t)^2 + \sin^2 t\right]}\ dt$$

$$a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\ dt$$

$$a\sqrt{2 - 2\cos t}\ dt$$

$$\sqrt{2}\, a \sqrt{1-\cos t}$$

Recall Half angle Identity

$$\sin\left(\frac{t}{2}\right) = \sqrt{\frac{1-\cos t}{2}}$$

or $\sqrt{1-\cos t} = \sqrt{2}\,\sin\left(\frac{t}{2}\right)$

$$\sqrt{2}\,a \int_0^{2\pi} \sqrt{2}\,\sin\left(\frac{t}{2}\right) dt = 2a\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$$

$$= \boxed{8a}$$

Arc length Formula

Let c be smooth curve represented by

$x = f(t)$; $y = g(t)$; $a \le t \le b$

$$S = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

ex $x = t^3 + 2$

$y = 2t^{9/2}$ $1 \le t \le 3$

$$S = \int_1^3 \sqrt{(3t^2)^2 + (9t^{7/2})^2}\, dt$$

$$\sqrt{9t^4 + 81t^7}$$

$$3t^2\sqrt{1 + 9t^3}$$

$$\int_1^3 3t^2 \sqrt{1 + 9t^3}\, dt$$; u = sub

$u = 1 + 9t^3$

$du = 27t^2\, dt$

$$\int_{10}^{244} \sqrt{u}\, \frac{du}{9} = \frac{1}{9} \int_{10}^{244} u^{1/2}\, du$$

$$\frac{1}{9} \left.\frac{u^{3/2}}{3/2}\right|_{10}^{244} \qquad \frac{2}{27}\left[244^{3/2} - 10^{3/2}\right]$$

$$\frac{4}{27}\left[244\sqrt{61} - 5\sqrt{10}\right]$$

* Finding slope of the tangent line of a cycloid:

$$x = a(t - \sin t) \; ; \; y = a(1 - \cos t) \; ; \; 0 < t < 2\pi$$

$$a > 0$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a - a\cos t} = \frac{\sin t}{1 - \cos t}$$

note: Horizontal Tangents (points)

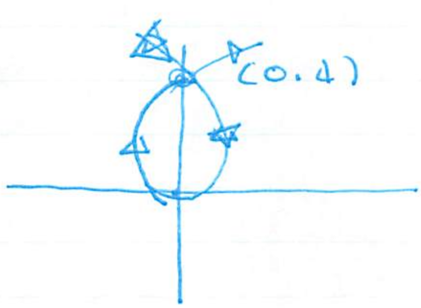
$$\sin(t) = 0 \; ; \; (t = \pi) \; ; \; \frac{dx}{dt} \neq 0$$

note The equation when $t = \pi$ $(\overset{x}{\pi a}, \overset{y}{2a})$,

$$\boxed{y = 2a}$$

* Finding an Equation of the tangent line to a Smooth Curve

$$x = t^3 - 4t \; ; \; y = t^2$$



what t corresponds to $(0, 4)$?

$$0 = t^3 - 4t \; ; \; 4 = t^2$$

$$t(t^2 - 4) = 0 \qquad t^2 - 4 = 0$$

$$\cancel{t(t+2)(t-2) = 0}$$

$$t = 0, \; t = -2, \; t = 2 \qquad t = \pm 2$$

$$\frac{dy}{dt} = \frac{2t}{3t^2 - 4}$$

exclude $t = 0$!

$$t = \pm 2$$

$t = -2$; $m = -\frac{1}{2}$

$$y = -\frac{1}{2}x + 4$$

$t = 2$; $m = \frac{1}{2}$

$$y = \frac{1}{2}x + 4$$