Improper Integrals

In order to use The Fundamental Theorem of Calculus the integrand needs to be continuous over a closed and bounded interval $[a, b]$. Then we have the following.

$$\int_a^b f(x)dx = F(b) - F(a)$$

The question now is what happens when we do not satisfy the original assumptions of the function being **continuous** or the interval we have an **infinite interval**? The answer is we have an improper Integral. Again, we need to consider cases and types.

Type 1 Infinite Intervals $[a, \infty)$ or $(-\infty, b]$ or $(-\infty, \infty)$

We still require that the integrand be a continuous function over the interval. Then we will have the following definitions.

Case 1 $[a, \infty)$

Def $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ for all $a < t$

Def- we say the improper integral $\int_a^\infty f(x)dx$ **converges**,
if the limit $\lim_{t\to\infty} \int_a^t f(x)dx$ exists and is finite (a number).

Notation $\int_a^\infty f(x)dx < \infty$

Def- we say the improper integral $\int_a^\infty f(x)dx$ **diverges**,
if the limit $\lim_{t\to\infty} \int_a^t f(x)dx$ does not exist or is infinite.

Notation $\int_a^\infty f(x)dx = \infty$

Case 2 $(-\infty, b]$

Def $\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)dx$ for all $t < b$

Def- we say the improper integral $\int_{-\infty}^{b} f(x)dx$ **converges**,
if the limit $\lim_{t\to-\infty} \int_{t}^{b} f(x)dx$ exists and is finite (a number).

Notation $\int_{-\infty}^{b} f(x)dx < \infty$

Def- we say the improper integral $\int_{-\infty}^{b} f(x)dx$ **diverges**,
if the limit $\lim_{t\to-\infty} \int_{t}^{b} f(x)dx$ does not exist or is infinite.

Notation $\int_{-\infty}^{b} f(x)dx = \infty$

Case 3 $(-\infty, \infty)$

Def $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$ for some c in $(-\infty, \infty)$

Def- we say the improper integral $\int_{-\infty}^{\infty} f(x)dx$ **converges**,
if both improper integrals $\int_{-\infty}^{c} f(x)dx \ and \ \int_{c}^{\infty} f(x)dx$ converge.

Notation $\int_{-\infty}^{\infty} f(x)dx < \infty$

Def- we say the improper integral $\int_{-\infty}^{\infty} f(x)dx$ **diverges**,
if at least one improper integral $\int_{-\infty}^{c} f(x)dx \ or \ \int_{c}^{\infty} f(x)dx$ diverge**.**

Notation $\int_{-\infty}^{\infty} f(x)dx = \infty$

Type 2 Discontinuous Integrands

We have a point in a closed and bounded interval $[a, b]$ where our function is discontinuous. That is, $f(x)$ is discontinuous for some c in $[a, b]$

Case 1 The function $f(x)$ is continuous over $[a, b)$ and is discontinuous at the right endpoint b.

Def $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$ for $a < t < b$

If the limit exists and is finite we have **convergence**. Else, we have **divergence**.

Notation $\int_a^b f(x)dx < \infty$ for convergence; $\int_a^b f(x)dx = \infty$ for divergence.

Case 2 The function $f(x)$ is continuous over $(a, b]$ and is discontinuous at the left endpoint a.

Def $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$ for $a < t < b$

If the limit exists and is finite we have **convergence**. Else, we have **divergence**.

Notation $\int_a^b f(x)dx < \infty$ for convergence; $\int_a^b f(x)dx = \infty$ for divergence.

Case 3 The function $f(x)$ has a discontinuity at some c in $[a, b]$ that is not at an endpoint.

Def $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $a < c < b$

If both improper integrals $\int_a^c f(x)dx \text{ and } \int_c^b f(x)dx$ converge, then the improper integral $\int_a^b f(x)dx$ **converges**. Else, we have **divergence**.

Notation $\int_a^b f(x)dx < \infty$ for convergence; $\int_a^b f(x)dx = \infty$ for divergence.