# Differentiability

**Def**- We say a function is differentiable at $x = a$, if $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

Note- $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

**Def-** We say f is differentiable on an open interval $I$ if it is differentiable for every number in the interval $I$

Note- I can be $(a, b)$ or $(a, \infty)$ or $(-\infty, b)$

**Theorem**
If f is differentiable at $x = a$, then f is continuous at $x = a$

**Proof** Assume f is differentiable at $x = a$

Then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

So $f(x) - f(a) = \frac{f(x) - f(a)}{x - a} x - a$

$\to \lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left[ \frac{f(x) - f(a)}{x - a} x - a \ \right] = \lim_{x \to a} \left[ \frac{f(x) - f(a)}{x - a} \right] \lim_{x \to a} [x - a] = f'(a) \cdot 0 = 0$

$$\to \lim_{x \to a} [f(x) - f(a)] \to \lim_{x \to a} [f(x)] - \lim_{x \to a} [f(a)] = 0$$

$$\to \lim_{x \to a} [f(x)] - f(a) = 0$$

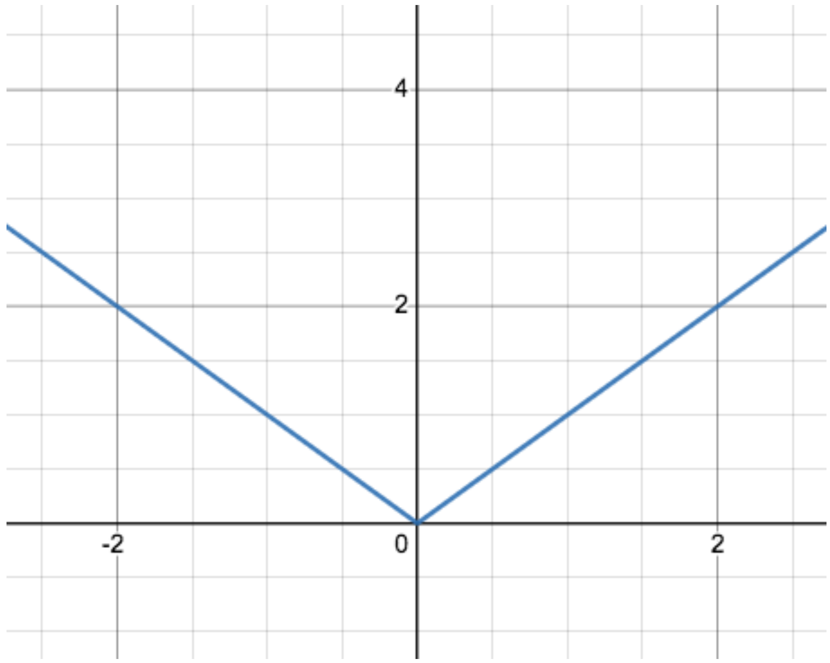
$$\to \lim_{x \to a} [f(x)] = f(a)$$

Thus f is continuous at $x = a$ ■

**Contrapositive is Valid**
If f is discontinuous at $x = a$, then f is not differentiable at $x = a$

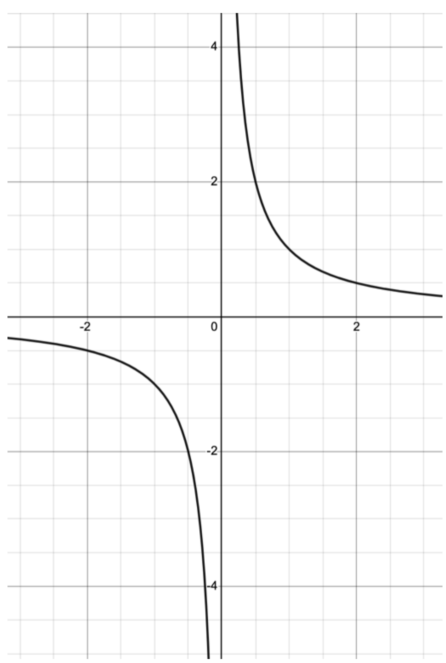
**Converse is Invalid**

If a function is continuous at $x = a$, then f is differentiable at $x = a$

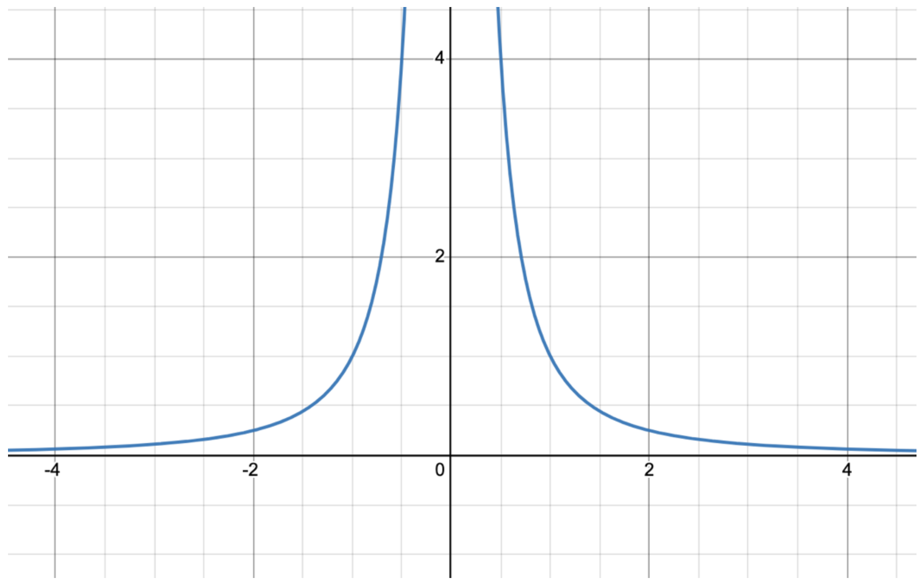



**Vertical Asymptote**

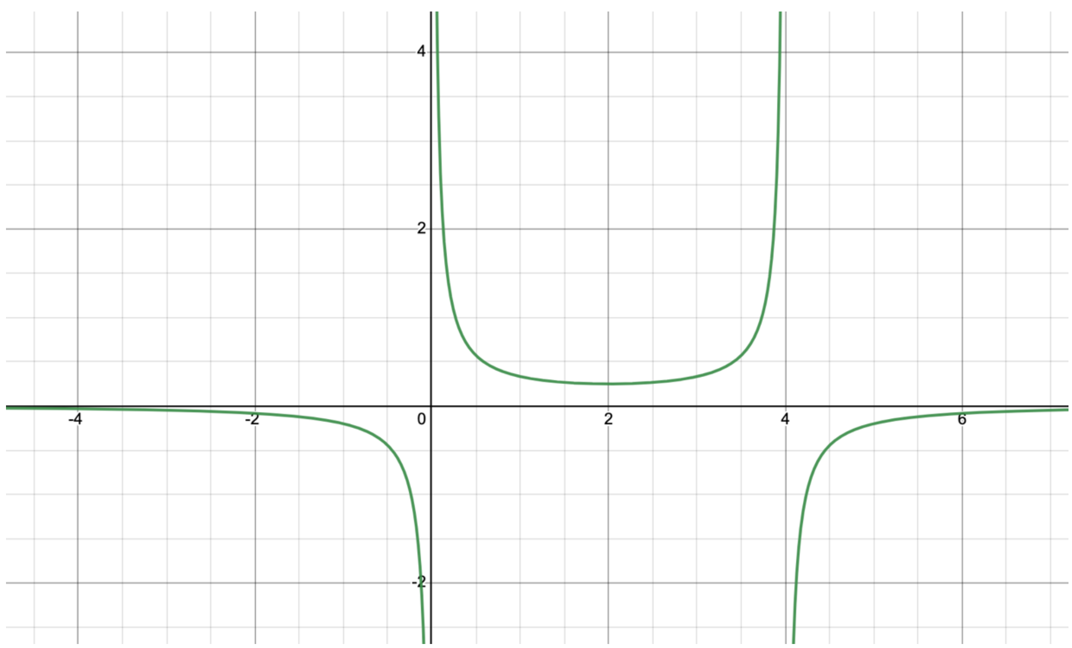
$$f(x) = \frac{1}{x}$$



$$f(x) = \frac{1}{x^2}$$



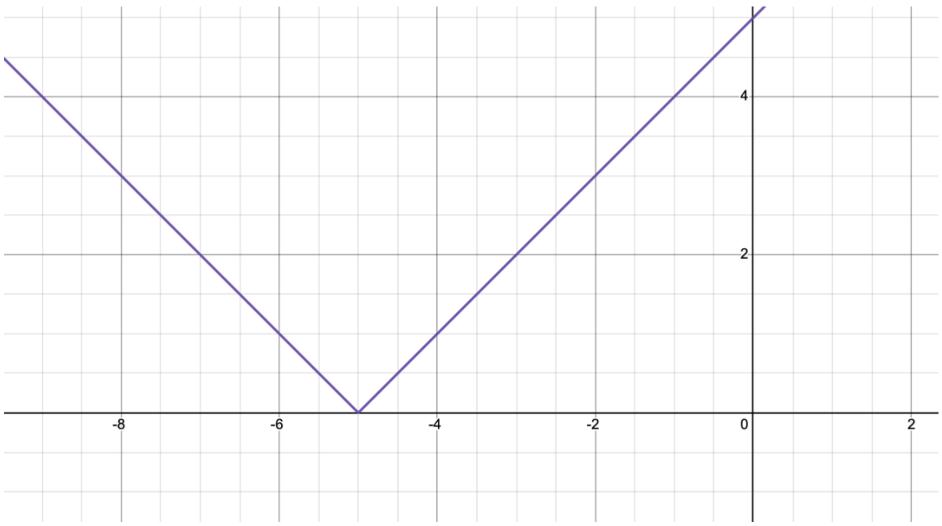

$$f(x) = -\frac{1}{x(x-4)}$$



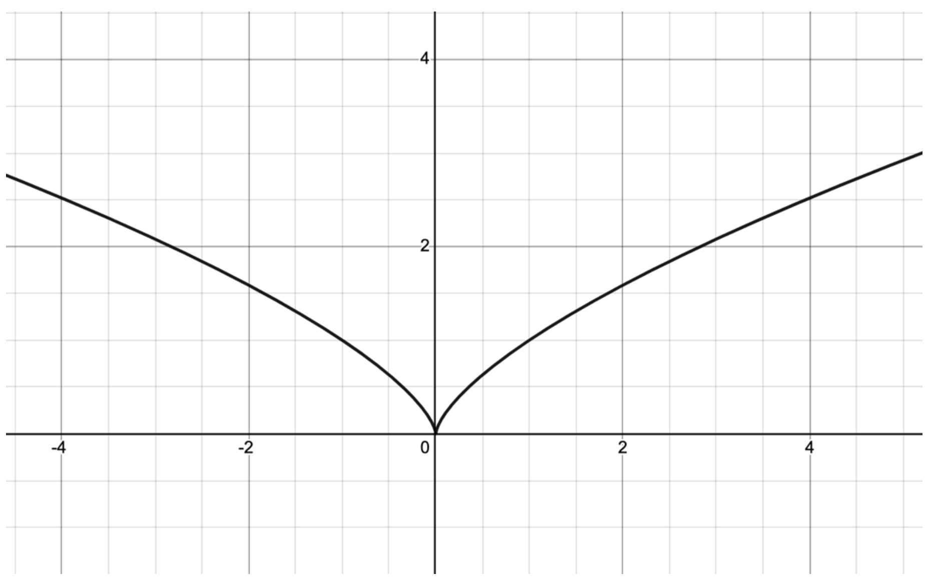
**Additional Circumstances of Non-Differentiability**

**Corner**

$$f(x) = |x + 5|$$




$$f(x) = x^{2/3}$$



**Jump Discontinuity**

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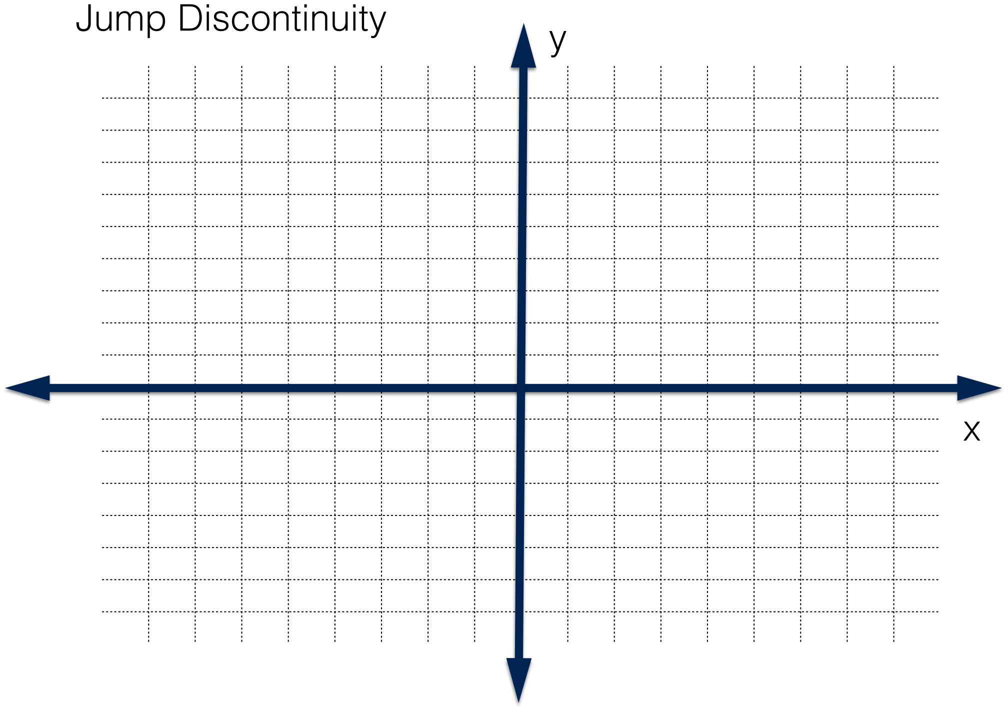
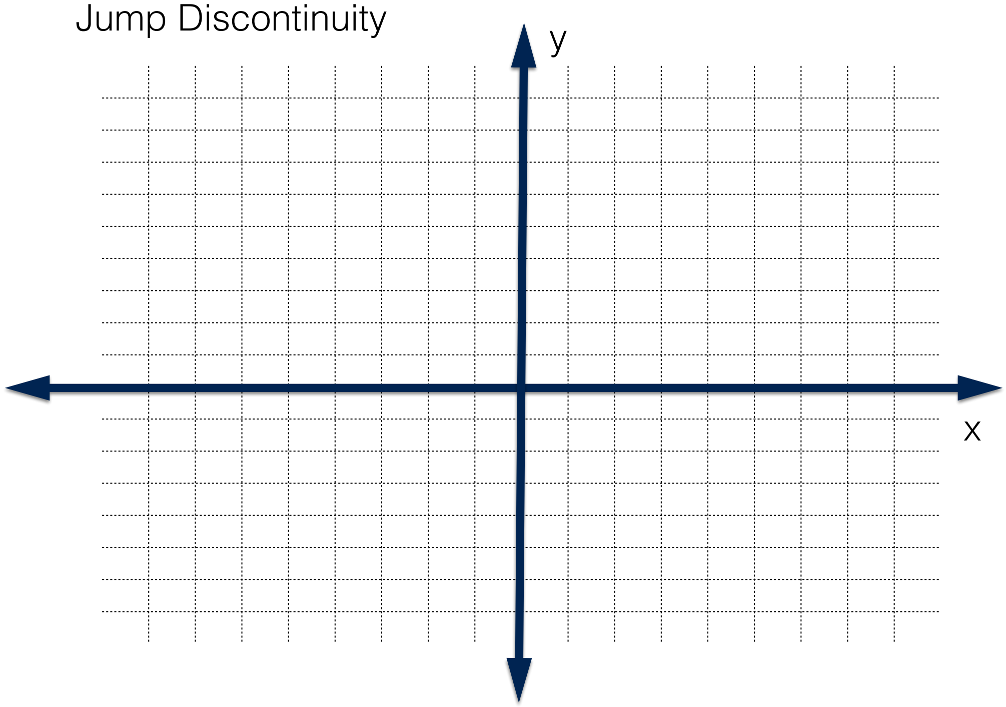
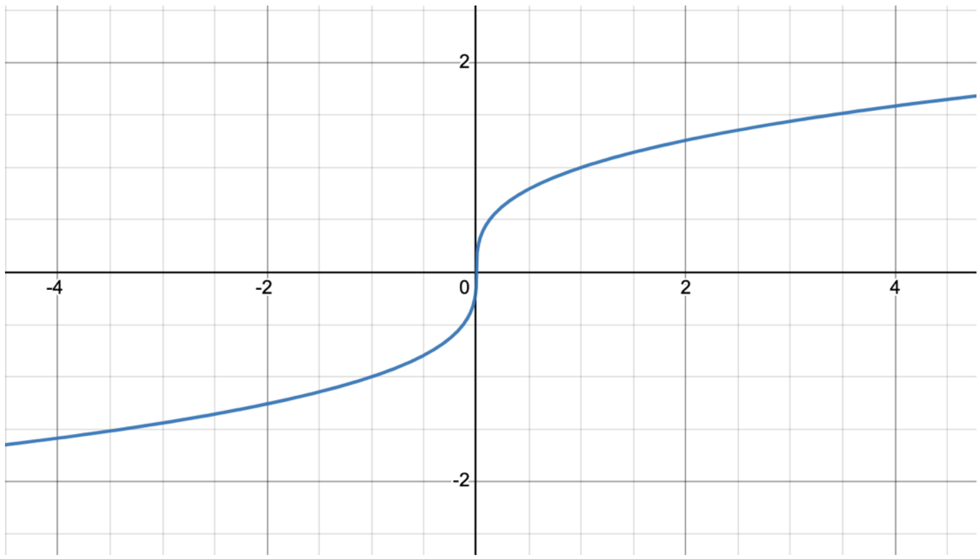



Jump Discontinuity



**Vertical Tangent**

$$f(x) = \sqrt[3]{x}$$




The function f is continuous at $x = a$ but has a vertical tangent line at $x = a$

$$\lim_{x \to a} |f'(x)| = \infty$$