Continuous Functions

Q What does it mean for a function to be continuous at a point?

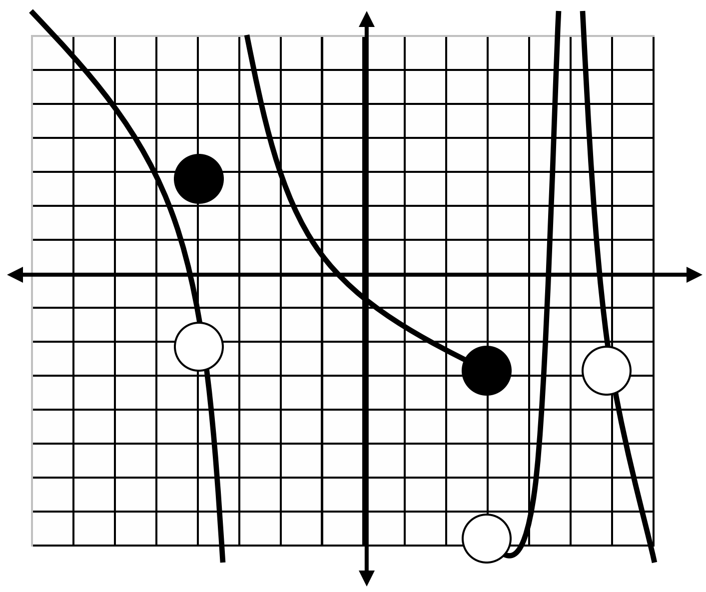
Answer- In mathematics, we have a definition that consists of three concepts that are linked in a special way. Consider the following definition.

Definition- Let P= $\left(a, f\left(a\right)\right)$ be a point on the curve f.
We say a function **f is continuous at a point P** if and only if the following relationship is satisfied.

$$\lim_{x\to a} f(x) = f(a)$$

The limit exists and is equal to the value of the function at a.

Consider the following curve and answer the questions below.



a	$\lim_{x\to a^-} \quad x)$	$\lim_{x\to a^+} \quad x)$	$\lim_{x\to a} \quad x)$	$f(a)$	Continuous	**Summary**
-6	5	5	5	5	Yes	f is continuous at x=-6.
-4	-2	-2	-2	2	No	f is discontinuous at x = -4.
-3	$-\infty$		DNE	Und	No	f is discontinuous at x = -3
-2	3	3	3	3	Yes	f is continuous at x = -2
3	-3	-8	DNE	-3	No	f is discontinuous at x = 3
5		$+\infty$		Und	No	f is discontinuous at x = 5
6	-3	-3	-3	Und	No	f is discontinuous at x = 6

Q What does it mean for a function to be continuous" everywhere"?

Answer-A continuous "everywhere" function (aka a continuous function) is a function that has NO points of discontinuity. The previous example is not a continuous "everywhere" function as it has discontinuities at x=-4, x=-3, x=3, x=5, and x=6.

Definition- Let I be an open interval.
We say f is a continuous on I, if f is a continuous for every point in I.

Example- Consider the function above and the open interval (-2,2).
The function f has no discontinuities in the interval (-2,2), thus we can say f is continuous over the interval (-2,2). But, it is not a continuous “everywhere” function as there exists points of discontinuity.

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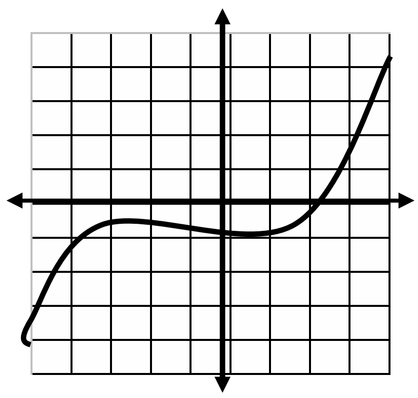
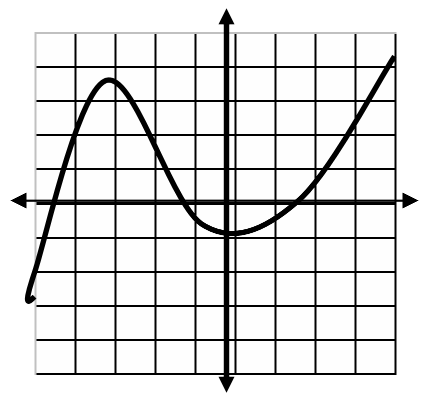
Q What does a continuous” everywhere” functions look like?

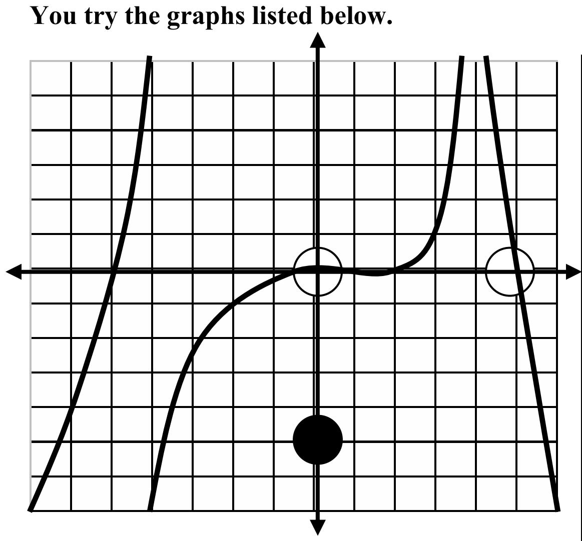
Answer- Any function that has no points of discontinuity, and there are many.





Intuitively, continuous everywhere functions are functions that can be drawn without having to lift up a pencil!

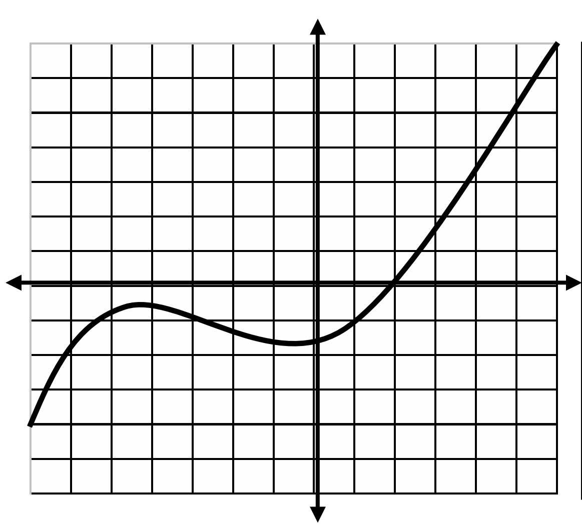
You try the graphs listed below.



a	$\lim_{x \to a} f(x)$	$f(a)$	Continuous
-6			
-4			
-2			
0			
3			
4			
5			

Note- In order to draw this curve you must lift up your pencil.

This is a discontinuous function.



a	$\lim_{x \to a} f(x)$	$f(a)$	Continuous
-6			
-3			
1			
2			
5			

Note- In order to draw this curve you do not have to lift up your pencil.

This is a continuous everywhere function.

Properties of Continuous Functions

Let f and g be two continuous at x=a, and let c be a constant, Then :

- cf is continuous at a (*see proof in class*).
- $f+g$ is continuous at a (*see proof in class*).
- $f-g$ is continuous at a (*see proof in class*).
- $f \cdot g$ is continuous at a (*see proof in class*).
- $\frac{f}{g}$ is continuous at a, if $g(a) \neq 0$ (*see proof in class*).

These properties are needed to prove the following facts.

Fact- The constant function $f(x)=c.$ is continuous.

Proof- Let c be some constant such that $f(x)=c.$ and a an arbitrary value.

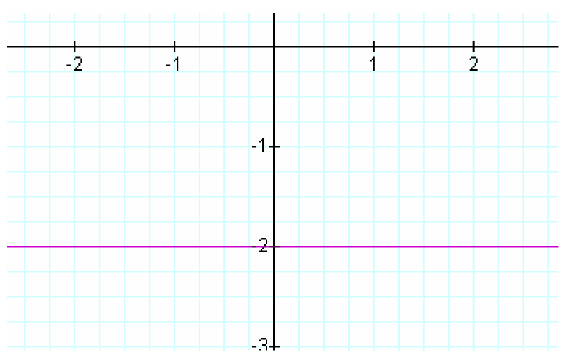
Then $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$ by properties of limits.

But $f(a)=c$ as f is the constant function.

Thus $\lim_{x\to a} f(x) = f(a)$ and f is continuous at a.

Since a is arbitrary, f is continuous everywhere □

Example- $f(x)=-2$



Fact- The identity function $f(x)=x$ is continuous.

Proof: Let $f(x)=x$ be the identity function and a some arbitrary value.

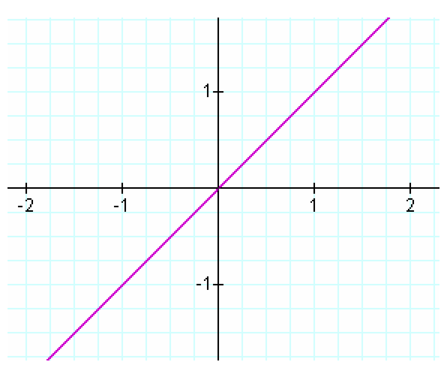
$\lim_{x\to a} f(x) = \lim_{x\to a} x = a$ by the definition of a limit.

But $f(a)=a$ as f is the identity function,

Thus $\lim_{x\to a} f(x) = f(a)$ and f is continuous at a.

Since a is arbitrary, f is continuous everywhere □

Graph



Fact- The function $f(x)=x^2$ is continuous.

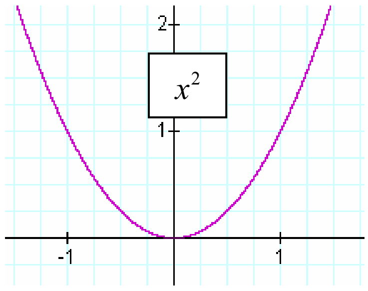
Proof: Let $f(x)=x^2$ and a an arbitrary value.

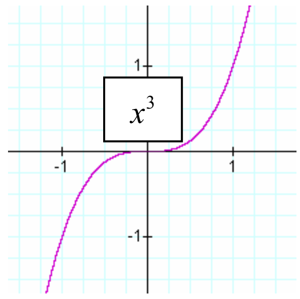
Then $\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = \lim_{x\to a} x \cdot x = \lim_{x\to a} x \cdot \lim_{x\to a} x = a \cdot a = a^2$ by properties of continues functions.

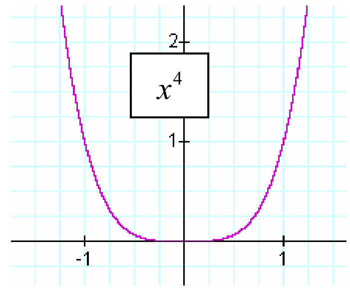
But $f(a)=a^2$, hence $\lim_{x\to a} f(x) = f(a)$.

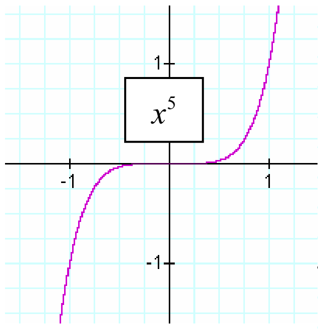
Since a is arbitrary, f is continuous everywhere □

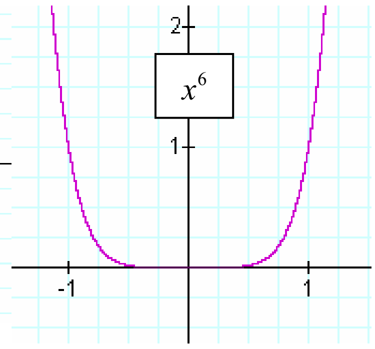
This can be generalized to show that $f(x)=x^n$ for any natural number n is continuous everywhere.

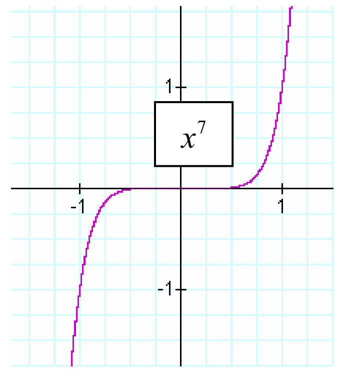












Fact- Polynomial functions are continuous.

Proof: Let P be a polynomial function such that $P(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1+a_0$ for some natural number n, and a an arbitrary number.
Then

$$\lim_{x\to a} p(x)=\lim_{x\to a}\left(a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1 x+a_0\right)=\lim_{x\to a}\left(a_n x^n\right)+\lim_{x\to a}\left(a_{n-1}x^{n-1}\right)+\ldots+\lim_{x\to a}\left(a_1\right)+\lim_{x\to a}\left(a_0\right)$$

$$=a_n\lim_{x\to a}\left(x^n\right)+a_{n-1}\lim_{x\to a}\left(x^{n-1}\right)+\ldots+a_1\lim_{x\to a}\left(x\right)+\lim_{x\to a}\left(a_0\right)$$ by properties of limits.

$$=a_n a^n + a_{n-1}a^{n-1}+\ldots+a_1 a+a_0$$ as x^n is continuous for all n.

But $P(a)==a_n a^n + a_{n-1}a^{n-1}+\ldots+a_1 a+a_0$, hence $\lim_{x\to a} P(x)=P(a)$.

Since a is arbitrary, P is continuous everywhere □

Fact- Rational functions are continuous.

Proof: Let R be a rational function such that $R(x)=\dfrac{P(x)}{Q(x)}$ where P and Q are polynomials, a an arbitrary number such that $Q(a)\neq 0$

Then $\lim_{x\to a} R(x)=\lim_{x\to a}\dfrac{P(x)}{Q(x)}=\dfrac{\lim_{x\to a} P(x)}{\lim_{x\to a} Q(x)}=\dfrac{P(a)}{Q(a)}$ by properties of limits and the fact that polynomial functions are continuous everywhere.

But $R(a)=\dfrac{P(a)}{Q(a)}$, hence $\lim_{x\to a} R(x)=R(a)$.

Since a is arbitrary, R is continuous everywhere □

Fact- If g is continuous and $f(x)=\sqrt[n]{g(x)}$, then f is continuous for $n>0$.

Proof: Let f be continuous and a an arbitrary value.

Then $\lim_{x\to a} f(x)=\lim_{x\to a}\sqrt[n]{g(x)}=\sqrt[n]{\lim_{x\to a} g(x)}=\sqrt[n]{g(a)}=f(a)$ by properties of limits, and the fact that g is a continuous function.
Since a is arbitrary, f is continuous everywhere □

Fact- If $h(x)=f\left[g(x)\right]$ such that f is continuous at b and $b=\lim_{x\to a} g(x)$, then h is continuous at a.

That is, $\lim_{x\to a} h(x)=h(a)$; $\lim_{x\to a} f\left[g(x)\right]=f\left[g(a)\right]$; $\lim_{x\to a} f\left[g(x)\right]=f\left[\lim_{x\to a} g(x)\right]$

Proof: *See proof in class.*

Fact- If $h(x)=f\left[g(x)\right]$ such that g is continuous at a, and f is continuous at g(a), then h is continuous at a.

That is, a continuous function of a continuous function is a continuous function.

Proof: *See proof in class.*

Example- Let f and g be continuous functions with f(2)=-7 and $\lim_{x\to 2}[3f(x)-2g(x)]=6$, what is g(3)?

Use properties of continuity.

$$\lim_{x\to 2}[3f(x)-2g(x)]=\lim_{x\to 2}\left[3f\left(x\right)\right]-\lim_{x\to 2}\left[2g\left(x\right)\right]=3\lim_{x\to 2}\left[f\left(x\right)\right]-2\lim_{x\to 2}\left[g\left(x\right)\right]=6$$

by properties of continuity.

$$=3f\left(2\right)-2g\left(2\right)=3\cdot 7-2g\left(2\right)=21-2g\left(2\right)=6$$

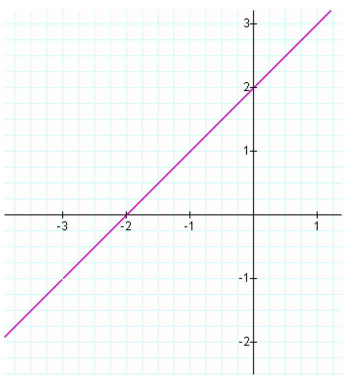
by the definition of continuity. Using algebra, g(2)=-15/2.

Example- Use continuity to evaluate the following limit $\lim_{x\to\pi}\cos\left[\cos\left(x\right)-x\right]$.

$$\lim_{x\to\frac{\pi}{2}}\left\{\cos\left[\cos\left(x\right)-x\right]\right\}=\cos\left\{\lim_{x\to\frac{\pi}{2}}\left[\cos\left(x\right)-x\right]\right\}=\cos\left\{\lim_{x\to\frac{\pi}{2}}\left[\cos\left(x\right)\right]-\lim_{x\to\frac{\pi}{2}}x\right\}$$

$$=\cos\left(0-\frac{\pi}{2}\right)=\cos\left(-\frac{\pi}{2}\right)=-\cos\left(\frac{\pi}{2}\right)=-1\cdot 0=0$$

Example- Explain why the function $f(x)=\dfrac{x^2+5x+6}{x+3}$ is discontinuous at x=-3.



Upon first glance, one notices a rational function, which may be require more care when graphing. However, when you simplify this rational function a linear function is revealed.

$$f(x)=\frac{x^2+5x+6}{x+3}=\frac{\left(x+3\right)\left(x+2\right)}{x+3}=x+2$$

This is the reason you see the graph of a linear function. However, closer look at the line representing this graph a **HOLE** at x=-3 is revealed.

Example- Determine whether the following function f is continuous.

$$f(x)=\begin{cases}\dfrac{x^2+5x+6}{x+3} & for\ x\neq -3\\ 4 & for\ x=-3\end{cases}$$

Using the definition of continuity, $\lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)=f(a)$

- When $x\neq -3$, we have a **Rational** function. We know **Rational** functions are continuous everywhere they are defined. Therefore, the top rational function is continuous except at –3.

- When $x=-3$, we have a **constant** function. We know **constant** functions are continuous everywhere.

But to use the definition, $\lim_{x\to -3^-}f\left(x\right)=\lim_{x\to -3^+}f\left(x\right)=-1$ while $f\left(-3\right)=4$

We can conclude that f is discontinuous at x=-3.

Example- Determine whether the following function f is continuous.

$$f\left(x\right)=\begin{cases}x^2-3x & x>3\\ 0 & -3\leq x\leq 3\\ -x+3 & x<-3\end{cases}$$

Using the definition of continuity, $\lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)=f(a)$

- When $x>3$ we have a polynomial function which is continuous everywhere. Therefore, f is continuous for $x>3$.

- When $-3\leq x\leq 3$ we have a constant function which is continuous everywhere. Therefore, f is continuous for $-3\leq x\leq 3$.

- When $x<-3$ we have a polynomial function which is continuous everywhere. Therefore, f is continuous for $x<-3$

The only concerns we should have are at the points $x=-3$ **and** $x=3$**.**
By simply using the definitions, we can determine continuity.

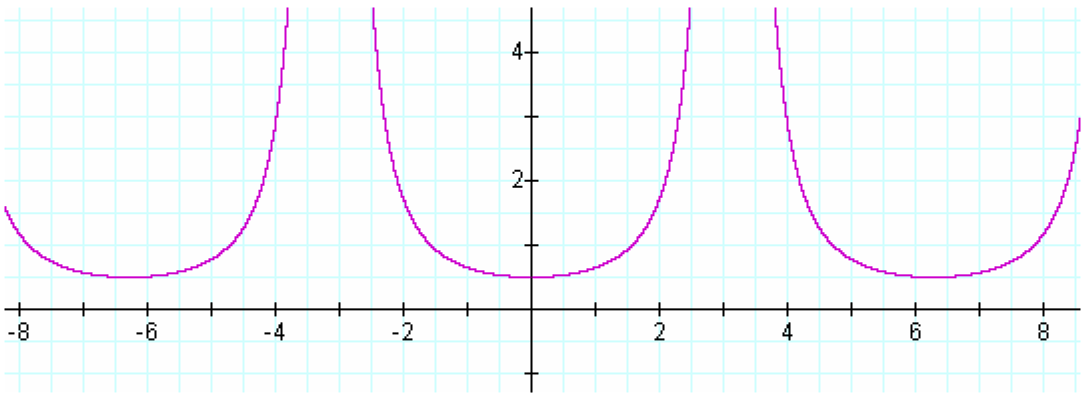
$\lim_{x\to -3} f(x)=6$ while $\lim_{x\to -3^{+}} f(x)=0$ so that f is discontinuous at x=-3.

$\lim_{x\to 3^{-}} f(x)=0$ while $\lim_{x\to 3} f(x)=0$, so that f is continuous at x=3.

Conclusion- The function f is continuous except at x=-3.

Example- Let $f(x)=\dfrac{1}{1+\cos x}$. Determine the values for which f is discontinuous.

The graph reveals various asymptotes.



We can determine these asymptotes by determining what values for x the denominator are 0.

$1+\cos x=0$ when $\cos x=-1$. This happens when $x=\pm\pi,\pm2\pi,\pm3\pi,\ldots$

Therefore, $\lim_{x\to\pi}\dfrac{1}{1+\cos x}=\lim_{x\to2\pi}\dfrac{1}{1+\cos x}=\lim_{x\to3\pi}\dfrac{1}{1+\cos x}=\ldots=\infty$

While, $\lim_{x\to-\pi}\dfrac{1}{1+\cos x}=\lim_{x\to-2\pi}\dfrac{1}{1+\cos x}=\lim_{x\to-3\pi}\dfrac{1}{1+\cos x}=\ldots=\infty$

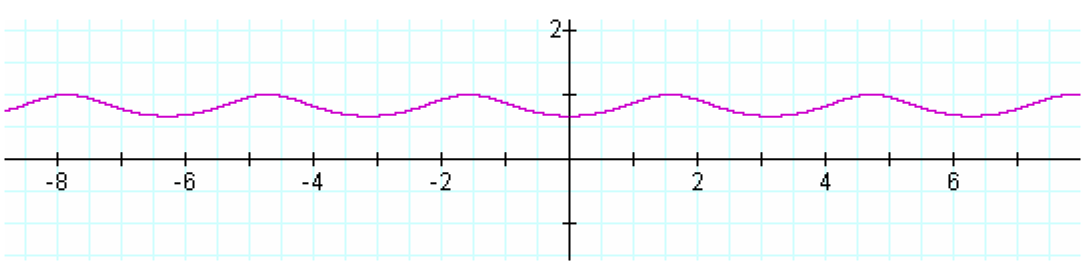
That is f is discontinuous at $x=\pm\pi,\pm2\pi,\pm3\pi,\ldots$

Example- Let $f(x)=\cos\{\sin[\cos(x)]\}$. State the domain of f.

- The domain of $\cos(x)$ is all real numbers.
- The domain of $\sin(x)$ is all real numbers

Therefore, the domain of $\sin[\cos(x)]$ is all real numbers

Thus, the domain of $\cos\{\sin[\cos(x)]\}$ is all real numbers



Intermediate Value Theorem
Let f be a continuous function over a closed interval [a,b].
Then there is a c in (a,b) such that f(a)<f(c)<f(b).

Example- Show that a root exists for the equation $x^3-3x+1=0$ in the interval $[0,1]$.

Let $f(x)=x^3-3x+1$

$f(0)=1$ which is a positive number also $f(1)=-1$ which is a negative number .

By the **Intermediate Value Theorem**, there exists a number c in $(0,1)$ such that $-1\le f(c)\le1$

That is, there is a c such that $f(c)=0$.