Bayes' Theorem

Aka, Bayes' Law or Bayes' Rule

Recall we derived this formula in the **Probability Multiple Selections Lecture** as we were looking into the proof of the Multiplication Rule for Probability.

Baye's Theorem

$$\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

This opens the door to a type of probability study known as **Bayesian Inference** which is very useful in determining conditional probabilities when we do not have a table of values to consider. The idea behind Bayes' Law is that we are measuring "a degree of belief" in a proposition A before and after accounting for the evidence called B. We have the following definitions.

Def $P(A)$ is known as the **Prior Probability** which represent the initial belief.

Def $P(A|B)$ is known as the **Posterior Probability** having accounted for the condition B.

The ratio $\boldsymbol{\frac{P(B|A)}{P(B)}}$ represents the **support B provides A**.

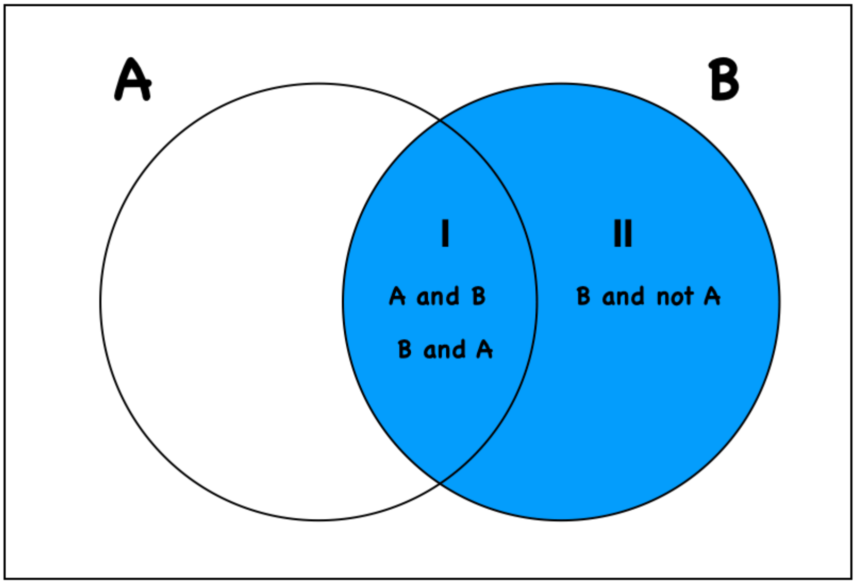
The condition B is assumed to be "fixed" and the event A is allowed to vary do that we can determine "a degree of belief" which is a probability.

We have an alternative form for **Baye's Theorem** that requires some Mathematical proof as well.

Alternative Form

$$\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}}$$

This proof relies on the statement that $\boldsymbol{P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$ and we start by considering a **Venn Diagram**.



We can observe that the circle B colored in blue is the union of region I and region II. We can represent this as $B = R_I \cup R_{II} = (B \cap A) \cup (B \cap \bar{A})$ and determine the probability of B.

$$P(B) = P((B \cap A) \cup (B \cap \bar{A}))$$

$$= P(B \cap A) + P(B \cap \bar{A}) - P(Both)$$

$$= P(B \cap A) + P(B \cap \bar{A})$$

Region I and region II are mutually exclusive (no overlap) so $P(Both) = 0$ and

$$P(B) = P(B \cap A) + P(B \cap \bar{A})$$

$$= P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$

which provides us with a key relationship for the alternative form.

$$\boxed{\boldsymbol{P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}}$$

This is how we can start to use **Bayes' Theorem** in the most famous settings of false positives and false negatives.

Cat Allergy?

Carmen is feeling itchy. There is a test for people who may be allergic to cats, however the test is not always right.

- People who really **do** have the **allergy**, the test indicates yes (test +) 80% of the time.

$$P(test + | allergy) = 0.8$$

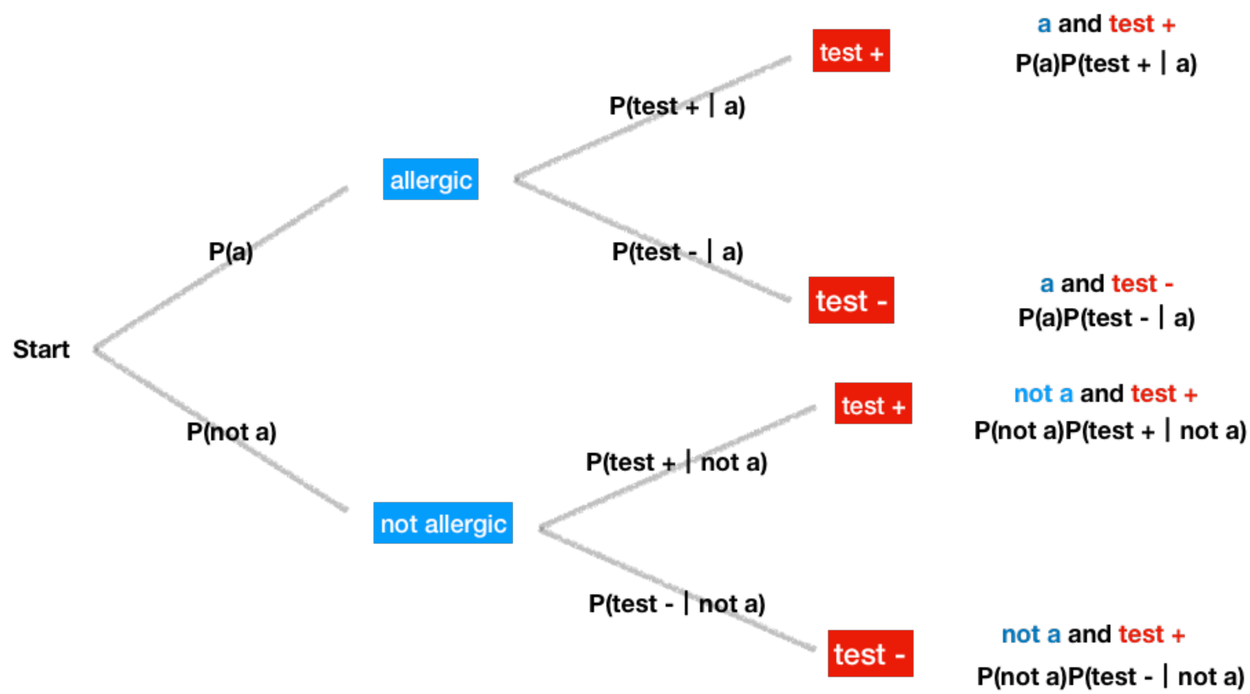
- People who **do not** have the **allergy**, the tests says yes (test +) 10% of the time, a false positive.

$$P(test + | not\ allergic) = 0.10$$

We know 1% of the population has the allergy. If Carmen's test says yes (test +), what is the chance that Carmen really has the allergy?

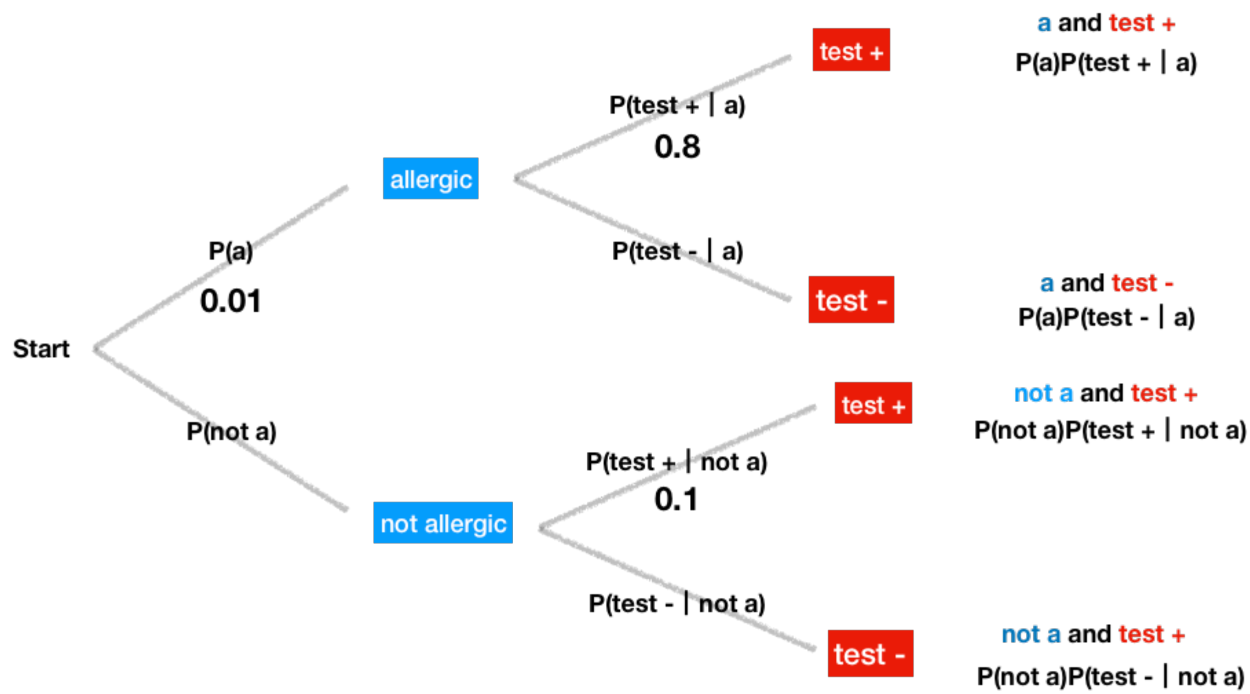
The additional information is $P(allergy) = 0.01$ and the question being asked is $P(allergic|test +)$?

To answer I would like to illustrate a tree diagram.

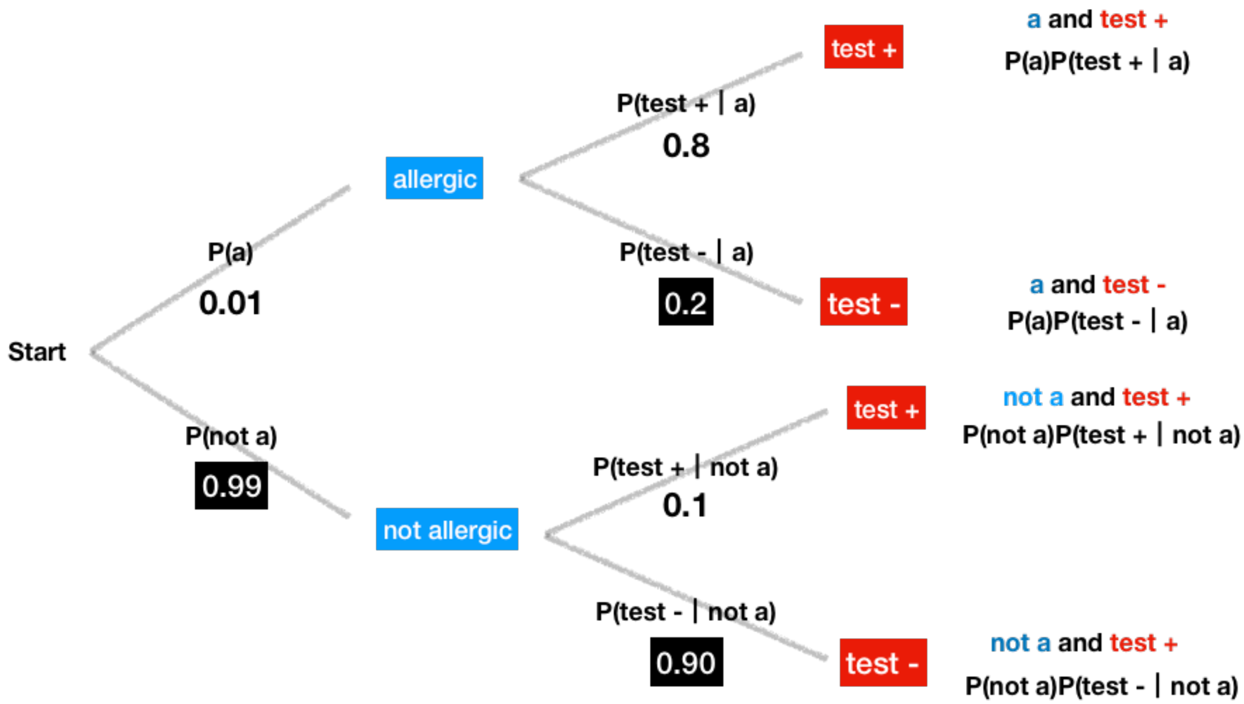


But, we need to fill in the three given values.

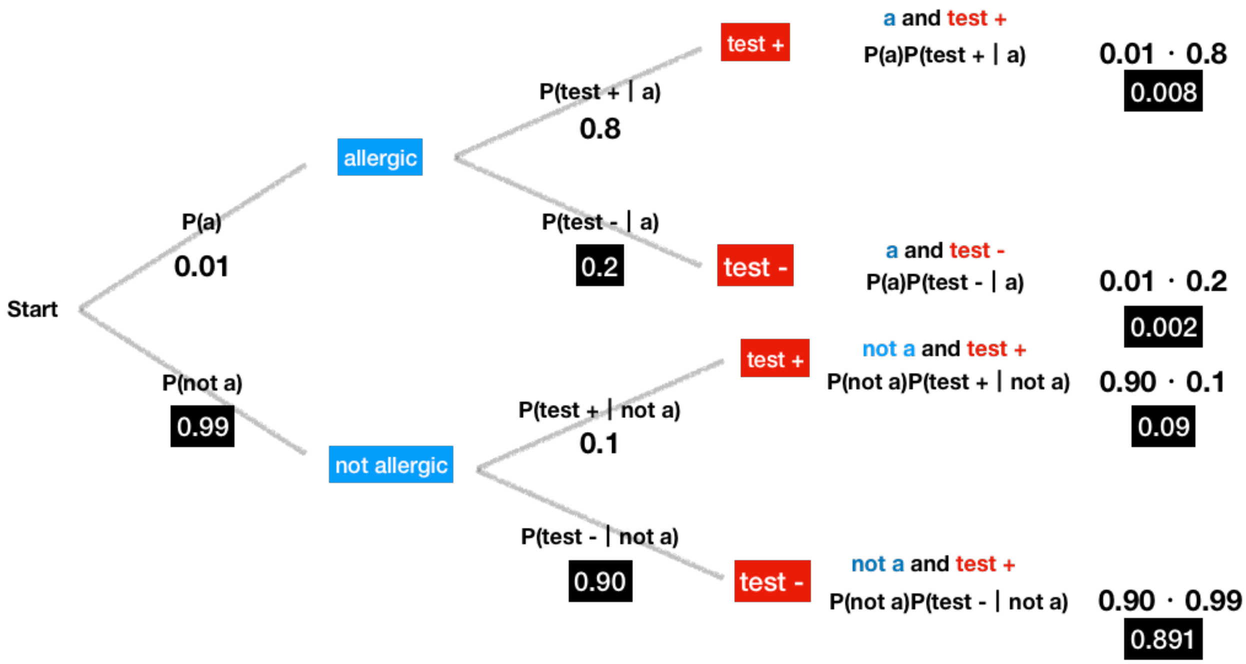
$$P(test + |allergy) = 0.8$$
$$P(test + |not\ allergic) = 0.10$$
$$P(allergy) = 0.01$$



Now, using the **Complement Rule for Probability** we can obtain the other missing values.



Follow the braches and obtain the remaining probabilities



With this information and **Bayes' Theorem**, we can now obtain answer the question.

$$\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

$$P(allergic|test+) = \frac{P(test+|allergic)P(allergic)}{P(test+)}$$

$$= \frac{0.8 \cdot 0.01}{P(test+)}$$

But, $P(test+) = P(test+|a)P(a) + P(test+|not\ a)P(not\ a)$

$$= 0.8 \cdot 0.1 + 0.1 \cdot 0.99$$

$$= 0.179$$

Since, $\boldsymbol{P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$ as we determined above.

We can now finally answer the question.

$$P(allergic|test+) = \frac{P(test+|allergic)P(allergic)}{P(test+)}$$

$$= \frac{0.8 \cdot 0.01}{P(test+)}$$

$$= \frac{0.8 \cdot 0.01}{0.179}$$

≈ 0.045 or 4.5% **false positive** rate

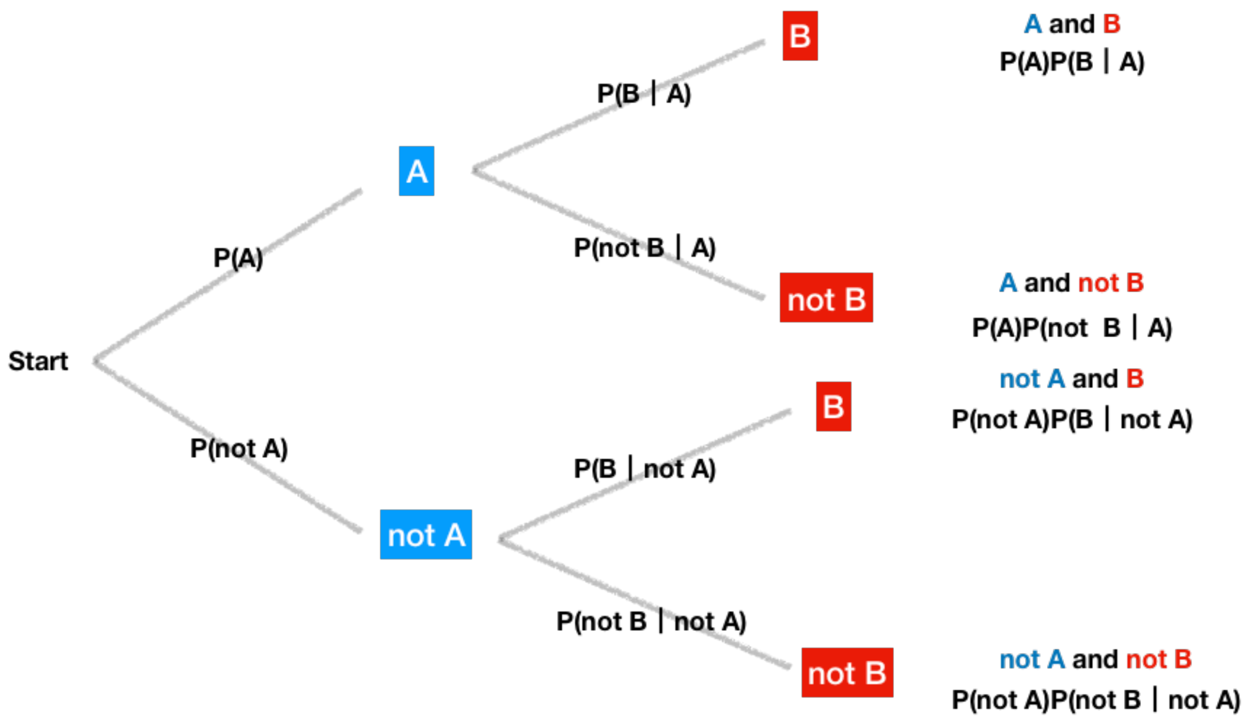
Generally speaking, **Bayes' Theorem** is applied by considering a **tree diagram.**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

and

$$P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$

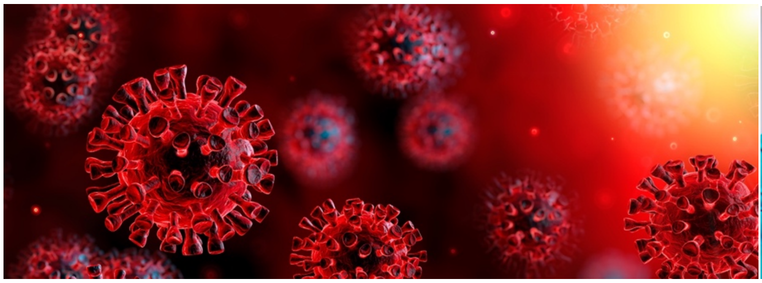
Tree Diagram



In short,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

In terms of **Diagnostic Testing** we can apply **Bayes' Theorem** which involves a few specific definitions.



This requires knowledge of a prior probability and then involves new information to update it. Here the **prior probability** is the doctor's belief whether the patient is infected or not. **The test result is used as information which changes the doctor's belief.**

Prior Probability $P(infected)$
Initial probability of infection based on the doctor's judgement.

Posterior Probability $P(infected|test+)$
The reliability of a positive result.

Posterior Probability $P(not\ infected|test-)$
The reliability of a negative result.

Bayes Theorem connects the doctor's initial belief with his/hers final belief.

$$\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

$$P(infected|test+) = \frac{P(test+|infected)P(infected)}{P(test+)}$$

and the denominator $\boldsymbol{P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$ is expressed

$$P(test+) = P(test+|inf)P(inf) + P(test+|not\ inf)P(not\ inf)$$

Together in the alternate form we have the following.

$P(inf|test+)$

$$= \frac{P(test+|inf)P(inf)}{P(test+|inf)P(inf) + P(test+|not\ inf)P(not\ inf)}$$

This gives rise to two unknown probabilities.

$\boldsymbol{P(test+|inf)}$ which should be high as the person is infected.

$\boldsymbol{P(test+|not\ inf)}$ which should be low as non-infected patients should be testing negative.

Fact- These two probabilities are actually measures of a **tests reliability** and are expressed as its **sensitivity** and **specificity**.

Def- Sensitivity $\boldsymbol{S_n = P(test+|inf)}$

- Probability of a positive test result given infection which should be close to 100%
- The test is “sensitive” to the presence of the Corona virus (COVID-19). If the Corona Virus is present, the test will detect it.

Def- Specificity $\boldsymbol{S_p = P(test-|not\ inf)}$

- Probability of a negative result given no infection which should be close to 100%
- The test is “specific” to the Corona Virus (COVID-19). If there is no Corona virus infection, the test will not detect anything and return a negative result.